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ABSTRACT

This paper discusses the question, what kinds of learning and new knowledge about content and children's thinking can an innovative curriculum program motivate? Discussion is restricted to whole number operations and how changing knowledge, beliefs, and practice as both an individual and collective problem might be conceived. To the extent that changes in practice are motivated by teachers' own concerns and questions about what they do, and to the extent that these changes incorporate or create the possibility for generative domain knowledge, they will result in self-sustaining good practices. (Contains 47 references.) (ASK)

Considerations of Systemic Change and Teachers' Knowledge of Students' Novel Strategies for Whole-Number Operations

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Ball (1996) has suggested that at its core, the problem of reforming mathematics education is a problem of teacher learning (see also McLaughlin & Oberman, 1996). There are many aspects to this problem, including at the most general level what teachers need to learn and how they might learn it. Domain knowledge is a special concern, not least because many elementary teachers are not well prepared mathematically (Ball, 1990; Kennedy, 1990; Ma, in press; Post, Harel, Behr & Lesh, 1991). The current reform movement requires an understanding of mathematics that is deeper and more extensive than what many teachers hold. Teachers have been put in the predicament of having to "teach more than they understand" (Floden, 1997, p. 13).

It is not just how much mathematics teachers know that is at issue. It is also the nature of that mathematics and how it supports and is supported by instructional activity focused on children's thinking. Many teachers, for example, have proficient working knowledge of mathematical procedures, but less knowledge of the deep structure (Skemp, 1982) that provides the motivation for those procedures. Teachers may know how to explain a procedure such as the partial products approach to multidigit multiplication, but not have a well articulated understanding of what a developmental trajectory for multidigit multiplication might be. Teachers may think of multiplication as repeated addition, with only tacit awareness of the range of possible situations from which multiplication could emerge.

Although many teachers feel acutely the results of inadequate mathematical preparation, the problem is not solely, or even largely, one of individual knowledge deficits. It is a systemic problem, whose solution will likely require massive resources of time, money, and intellectual energy (Elmore, 1996; Sarason, 1990). The professional development demands of the current reform movement are daunting when cast in terms of the numbers of districts, schools and teachers that want to reform their mathematics instruction. There is thus a need for solutions that take into account the scope of the problem, as well as its qualitative dimensions.

A partial solution may exist in the design of curriculum materials to support teacher learning in practice. Ball and Cohen (1996) point out that "not only are curriculum materials well-positioned to influence individual teachers' work but, unlike many other innovations, textbooks are already 'scaled up' and part of the routine of schools" (p. 6). Such materials could be seen as "one component of a systemic approach to professional development" (p. 7). While curriculum materials cannot alone provide teachers the knowledge needed to teach in new ways, they can provide leverage for changes in practice and hence learning. A particular question is what kinds of learning and new knowledge about content and children's thinking can an innovative curriculum program motivate? In particular, what basis for making new knowledge claims about mathematics and about students might the teaching of such a curriculum provide teachers? A further question, broached in conclusion in anticipation of a larger-scale study, is how can professional development interventions extend and deepen that knowledge?

To focus these questions for this paper, I restrict the discussion to whole-number operations, and how we might conceive changing knowledge, beliefs and practice as both an individual and collective problem. I will argue that to the extent that changes in practice are motivated by teachers' own concerns and questions about what they do (cf. Little, 1993), and to the extent that these changes incorporate or create the possibility for generative domain knowledge (Confrey, in progress; Stroup, in progress), they will result in self-sustaining good practices.

Problems of Changing Educational Practice

Many teachers think the elementary mathematics curriculum consists mainly of learning number facts and standard algorithms for addition, subtraction, multiplication and division (Porter, 1989). Although this conception of the curriculum is not surprising, given the content and organization of most commercial textbooks, it is an impoverished view of mathematics (cf., Hersh, 1997; Thurston, 1991). Yet, because many teachers value whole-number operations above other areas of mathematics, curricular innovations in this area may scale more easily. Further, there has been a great deal of research on children's thinking about whole-number operations that justifies suggested changes in the teaching of whole-number operations. There is growing evidence, for example, that children who learn to add and subtract by first learning the standard algorithms involving regrouping suffer some astonishing deficits in their understanding of base-ten number concepts (Kamii & Dominick, 1998). There is also a great deal of evidence that children can use their growing understanding of base-ten number concepts to invent viable, relatively efficient, and even powerful alternative procedures for whole-number operations without direct teaching of standard algorithms (Ambrose, Baek & Carpenter, in press; Carpenter, Franke, Jacobs, Fennema & Empson, 1998; Fuson, Wearne, Hiebert et al., 1997; Kamii, 1994). A focus on inventing conceptually sound and powerful alternative algorithms has the potential to introduce into the classroom mathematically legitimate activity (Hiebert, Carpenter, Fennema, et al., 1996). Although in some ways this shift is small in terms of content, it is an enormous shift in terms of practice (see also Stroup [in progress] and his discussion of generative activity).

Teachers have not found it especially easy to initiate and sustain change in their instruction to take children's strategies into account, for several reasons. First, the idea that there are alternative ways to add, subtract, multiply and divide is a revelation to many teachers who were socialized to believe that there was one way only to solve a given mathematics problem. A second reason is that alternative strategies for whole-number computations depend on conceptual knowledge that many teachers feel unsure about or lack. A third reason is that changing instruction to accommodate and encourage children's invented strategies involves, a substantial shift in practices that the teacher believes to be effective (Smith, 1996). A fourth reason is that most curriculum materials simply do not support this kind of teaching (U.S. Department of Education, 1996). Even in studies that have documented "substantial and worthwhile change" (Richardson, 1990; Carpenter, Fennema, Peterson, Chiang & Loef, 1989), not all teachers changed or sustained change (Knapp & Peterson, 1995).

The challenge is to find ways to initiate and institutionalize changes in ways that affect what Elmore (1996) refers to as "the core" of instructional practice. Central to the core are basic conceptions about what counts as subject-matter knowledge, definitions of the teacher's and students' roles in the transmission and generation of knowledge, and classroom and school-level participation structures for teaching and learning, among other things. There is, as Elmore (1996) points out, no shortage of ideas about how to define and organize the core. He continues,

> The problem, then, lies not in the supply of new ideas, but in the demand for them. That is, the primary problem of scale is understanding the conditions under which people working in schools seek new knowledge and actively use it to change the fundamental processes of schooling. (Elmore, 1996, p. 4)

Thus, Ball's (1996) suggestion that the problem of reforming mathematics education is primarily a problem of teacher learning needs a caveat. It is a problem of teacher learning conceived as rooted in practice, and defined in terms of changing forms of participation (Lave & Wenger, 1991; Lave, 1996).

Examples of Teachers' Knowledge and Issues Raised

Ball and Cohen (1996) have proposed that properly designed textbooks have the potential to help scale change. Research on children's thinking about multidigit operations provides a backdrop against which to pose questions about the nature and extent of what teachers might learn as they teach using such materials. To consider this question, examples have been culled from a set of interviews with teachers about what they learned implementing *Investigations in Number, Data, and Space* (1995-1998, TERC). Participants consisted of all the teachers in grades three, four and five at a single elementary school, located in a district that has adopted *Investigations* as its elementary mathematics program. The analysis is ongoing, and will serve as a pilot study for a larger scale study planned for the next three years.

The first example is drawn from responses to a question asking teachers to generate alternative strategies for multiplying 18 times 25. A second example is drawn from responses to a question about how teachers would respond to a student's novel strategy for dividing 144 by 8 that was based on an incorrect use of repeated halving. Together these examples raise questions about what knowledge of students' thinking we might expect teachers to have and use, how domain knowledge might help teachers interpret and respond to novel strategies, and ultimately, how to generate the knowledge most useful to teachers. Preliminary analysis of the interviews suggests that teachers learned a great deal about student thinking from teaching *Investigations,* and that what they learned includes values, beliefs, and knowledge. Further it appears that teachers' beliefs about what students were capable of changed quite readily as a result of teaching the new curriculum, and that new domain knowledge may be much more difficult to acquire through practice alone. These findings have implications for the design of professional development, which are discussed in a final section.

Example 1: Teachers' Knowledge of Multidigit Multiplication

Probe: "One goal of Investigations *is to get students to solve problems in many different ways. Suppose that you were teaching multidigit multiplication. What are at least three different strategies that children might use to solve 18 x 25?"* Previous research on teachers' understanding of the conceptual basis of the standard multiplication algorithm has documented fairly weak understanding (Ma, in press). In many cases, teachers' knowledge was limited to correct execution of the algorithm. Even those teachers who could explain why the partial products should be shifted a digit to the left with each digit by digit multiplication, did not appear flexible in their understanding. In contrast, the teachers who participated in the current study were able to generate a wide variety of alternative algorithms for multidigit multiplication. Most of them reported that they had learned these strategies as a direct result of teaching *Investigations,* from students as well as the curriculum.

Flexible understanding of multiplication. All but one of the 13 teachers were able to generate at least three strategies that children might use to multiply 18 x 25. Many of the strategies drew on conceptual features of the numbers and the operation of multiplication. For example, several teachers described a partial products approach, making use of the distributive property. Ms. Rojas[1] said:

> Okay, so I know that we do the 10 times 20, the 8 times 20 and then they would do the 10 times 5 and the 8 times 5. And if you ask them why they'll do that, they'll tell you, they know the word now is called landmark, at that time they didn't. They'll tell you that 5s and 10s and 20s are easier to work with... And even in some of the children who may mathematically be even at a lower stage, if 20 still seems even bigger for them, they'll break that down even smaller and do 10 times 10, 10 times 10 twice as opposed to the 10 times 20. And then again with the 8 times 20 they'd break it down to 8 times 10, 8 times 10. And so then they would turn around and problem solve this (see Figure 1). They would go ahead and multiply 10 times 20 and get the 200. And they'd do the 8, 2 times, ah, the 20 times 8, excuse me, would be 160. And then they would do 10 times 5, would get the 50, and the 8 times 5 would get 40. And then they would turn around and add 'em all up. Which is what I am going to attempt to do. And get 450.

Though in one sense Ms. Rojas's strategy is isomorphic to the steps in the standard algorithm for multiplication, in an important sense it is quite different. Her description is couched in terms that relate how children might actually work through this problem, and demonstrates a flexible understanding of multiplication as an operation.

Another kind of strategy involved what the teachers called *clusters,* a term used in *Investigations*. Ms. Huston, for example, described how students would work with a cluster of 10 25s. Students who knew that 10 25s was 250 might then create a tower to count up the rest of the way by 25s (Figure 2a); or they might use the cluster as the

[1] All teachers' names are pseudonyms.

basis to figure 20 25s, and count back 2 25s (Figure 2b). Other teachers described this latter strategy directly in terms of compensation (e.g., Figure 3). These strategies suggest that, in sharp contrast to previous research, the teachers understood multiplication as more than a computational procedure.

Sense of developmental trajectory. Knowledge of possible knowledge trajectories may be one of the most powerful tools that a teacher of mathematics can use (Simon, 1997). Teachers also mentioned how children's thinking of multiplication developed. For example, Mr. Jaimez noted that the strategy a child would use to multiply 18 x 25

> ...just depends on the level of the child. Some kids are at the level where they would have to count each individual one. Some kids would be able to skip count, say 25, they'd say, 'Well every 4 is going to be 100,' so they'd do that. Some kids might skip count every 2 for 50, depending on how comfortable they were with that.

Ms. Rojas, in her response above, indicated that some children who were not comfortable with 20 times 10 would be able to break that down to "10 times 10 twice." These teachers' observations are in accord with what has been documented about the development of children's multidigit strategies elsewhere (e.g., Ambrose, Baek, Carpenter, in press). Further, several of these observations extended the information provided by *Investigations,* and so suggest that at least some of the teachers were engaged in practical inquiry (Richardson, 1990). That is, at least some of the teachers continued to refine and elaborate their models of student thinking on the basis of what they learned from their students.

Example 2: Teachers' Responses to a Novel Division Strategy

Probe: A student was solving 144 ÷ 8. She said, "I know, I can just split it in half. So I will keep dividing by 2. I need to do that 4 times, since 2+2+2+2 is 8." As she talked, she wrote this (see Figure 4). How would you respond to this student?

This item was designed to investigate what models of division teachers used, and how teachers sorted out the difference between additive and multiplicative composition in division.[2] The strategy draws on a partitive conception of division, since the goal appears to be to split 144 into 8 groups (in contrast to a measurement or quotative interpretation, which would involve finding how many groups of 8). The difficulty that the fictional student is having is based on how she knows when her repeated halving has resulted in the correct number of groups.

Teachers generally found this item difficult. Many of them expressed a desire to think more about the problem, and they all reported that they had not seen a strategy like this one. Several teachers also expressed frustration at not being able to probe the fictional student further. This response is notable because it suggests that these teachers do not necessarily have curriculum scripts that are automatically engaged in problematic situations (Leinhardt, Putnam, Stein & Baxter, 1991); instead, their goal was to get more information about the student's thinking (Floden, 1997). Another notable aspect of teachers' responses was their willingness to admit to the student that they were not quite sure what was going on and would need to think about it more. For example, Ms. Seagreen, a first-year teacher said, "I would have to just tell her that, you know, I need more time to figure out what she's done." These kinds of responses leave a final interpretation of the strategy open, rather than closing it off, and allows the student, teacher and even other students to continue thinking about how to make sense of it. In this sense, they are potentially generative.

While teachers demonstrated a flexible and multifaceted understanding of multiplication, they did not appear to be drawing similarly on an explicit set of models of division to interpret the student's work. It was not unusual for a teacher to vacillate between partitive and measurement interpretations of division in the same explanation. For example, one teacher said that she would ask the student to solve the problem using counters of some kind: "I would ask her to take them, and divide them into eight groups, and see what she gets then..." Continuing, "I don't think she really understands what

[2] The item originated as an actual strategy used by Debbie Junk's fourth-grade daughter

she's doing when she's dividing. She's dividing into groups of *eight*" (teacher's emphasis in talking).

Most of the teachers who were able to make sense of the strategy within the constraints of the interview did so using more formal mathematical concepts instead of generative models of the domain. For example, Ms. Hill concluded that

> This is a valid way to do it ... if you're at the point where you can break down the number ... where you can say [if dividing by 12], 12 is 2 times 2 times 3. Yeah. Then it will work. [Solves 24 divided by 12 by dividing successively by 2, 2 then 3.] So if they understand factoring, then it's a valid way to do it, but if they're not at that point, then I would stop with, 'It didn't work.'

Other teachers approached it through thinking about the inverse operation. For example, Ms. Huston worked her way back up the chain of repeated halving by repeatedly doubling the divisors, and discovered that the net result was that the student had divided by 16 instead of 8. Other teachers suggested that the student check her answer by multiplying 8 times 9. (One teacher, however, noted that while other teachers would probably suggest that the student check her answer, he had learned that children do not understand fully how inverse operations relate, and so thought it would not be an effective pedagogical move.)

In the absence of generative domain knowledge to inform interpretations of the student's strategy, the teachers relied on formal subject-matter knowledge and general pedagogical moves. Teachers clearly valued and respected the student's strategy -- as evidenced by comments such as "this is fascinating that a student did that" -- and they reported they would respond with suggestions and questions to elicit further thinking or strategies. While these kinds of responses are viable pedagogical moves and do not require sophisticated domain knowledge, they also do not necessarily engage the potential for mathematical growth inherent in the strategy. Similarly, the use of more formal mathematical knowledge to make sense of the strategy did not accord with the origin of the strategy in notions of splitting and repeated halving. More formally organized

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to solve a division problem, and was adapted for purposes of the interview.

interpretations missed the mathematically viable aspects of the student's strategy. This example suggests that, not surprisingly, there are limits to what teachers might be expected to learn simply by virtue of implementing an innovative curriculum. However, the teachers' willingness to try to understand the strategy, rather than rejecting it outright as wrong, signals a potential space for continued mathematical dialogue.

Discussion: Implications for Professional Development

Lave (1996) has written that learners "constitute the working conditions for teaching rather than the other way around" (p. 159; see also McLaughlin, 1993). *Investigations* is essentially a curriculum that creates a new kind of student for teachers. Because students are the context that "matters most" to teachers, it appears this new kind of student motivates teachers towards new practices and conceptions. Teachers commented over and over again that one of the most important things they had learned from teaching *Investigations* was that students were capable of inventing a wide variety of strategies to solve problems. Perhaps the most astonishing result of the interviews is that every teacher noted some aspect of their newfound respect for children's mathematics capabilities. If nothing else, the teachers had clearly acquired a pedagogical disposition that acknowledged the existence of a multiplicity of strategies, and was founded on a high regard for children's thinking.

The more difficult task for teachers involves without question the shift in perspective required to hear and understand the mathematical significance of children's thinking. There is a great deal of intellectual work involved in making such a shift that goes beyond subject-matter knowledge and pedagogical content knowledge (see for example Duckworth, 1996; Confrey, 1991; Castro-Filho & Confrey, 1999). It is an ongoing process. Formal preparation in mathematics does not typically prepare teachers to engage with student thinking in this way (Callahan & Confrey, in progress). In fact several teachers commented ruefully on the shortcomings of their own mathematical preparation.

It appears that *Investigations* supported the development of some aspects of generative domain knowledge for the teachers in multiplication. One speculation about why this kind of learning might have been facilitated is that the design of the curriculum was founded, when possible, on research knowledge about the development of children's thinking. This attention to research allowed the curriculum writers to design tasks that elicited fairly predictable patterns of reasoning, and to alert teachers to the variety and meanings of those strategies. However, one thing that *Investigations* did not do (and, as a set of curriculum materials, could not be expected to do) was to prepare teachers to detect and build on the mathematical potential in students' novel strategies. Teaching *Investigations* did however, foster a desire to continue learning mathematics. At the end of the interview, Mr. Jaimez said, "What I would like to do is take some time, if I can, to spend some time looking at these problems and, I mean, I'd love to figure out some more about it."

A preliminary claim is that as a result of teaching *Investigations,* the teachers as a group evolved a set of concerns about mathematics teaching that were centered on students' thinking, and grounded in new knowledge about alternative conceptual trajectories through the elementary curriculum. They have acquired a predisposition to elicit strategies from children, and to expect a variety of responses. They further appear to believe quite strongly that they as well as their children are benefiting from this kind of approach to mathematics. In short, the ground has been laid for professional development that is rooted in their concerns and practice, but also aims for the kind of generative domain knowledge that represents good mathematics.

A Model of Professional Development

An important goal of professional development is to produce sustained changes in practice that have a positive bearing on student outcomes on a wide scale. An extension of this goal is to produce teachers who engage in practical inquiry in ways that lead to generative understanding of student learning and the domain of mathematics

(Richardson, 1990). In mathematics, one model of generative learning is characterized by a) the centrality of children's thinking to teachers' practical inquiry, b) specific knowledge of children's thinking, c) the use of a framework to organize knowledge of children's thinking, and d) adapting and creating knowledge about children's thinking (Franke, Carpenter, Levi & Fennema, 1998). This model suggests that focusing professional development on issues related to children's thinking about specific mathematics content can provide a compelling motivation for teacher learning.

However, many of the existing models of teacher change that address issues of content and/or student thinking are developmental, in that they focus on how individuals and their knowledge and beliefs change over time relative to an image of the knowledge and beliefs that would support standards-based instruction (e.g., Simon, 1997; Goldsmith & Schifter, 1997; Fennema, et al., 1996). The disadvantage of such models is that they do not take into account the local needs and problems of teachers and schools, nor the intellectual, social or emotional contexts of teaching (Lieberman, 1996; Little, 1993). Further, those models that do take these things into account (e.g., Fullan, 1991) tend to not grapple with issues of how teachers might acquire deeper knowledge of students' content thinking, or with issues of how to make change systemic. Planning for lasting teacher change and learning will require taking into account factors such as the communities in which teachers participate in their schools, the social organization of teachers' work, and institutionally mandated goals for children's learning. The model in Figure 5 takes these factors into account through a community-of-practice perspective (Wenger, 1998), while keeping the focus on issues central to the core of instruction (Elmore, 1996). Knowledge of children's thinking and mathematics emerges from instructional activity and is subjected to a cycle of refinement based on near and far-term student outcomes, in partnership with researchers (these ideas build on work done by Franke et al. [1998], and Sherin [1998]).

One factor that was taken as a given in the current study is the role of district mandated curriculum in mathematics. One facet of the curriculum is a set of state specified goals in mathematics learning (Texas Essential Knowledge and Skills [TEKS]) that are tested every year beginning in third grade. For now I will not comment on the possible role of these objectives in teaching and learning; the relationship to practice and outcomes is complex. A second facet is the fact that the district adopted *Investigations* as a result of procuring a Local Systemic Change grant from the National Science Foundation. All teachers were consequently expected to implement the program. The teachers at the school where the interviews were conducted had all essentially bought in to this curriculum; they had agreed to try it out, some with more skepticism than others. It is the fact of buy-in that I would like to highlight now as being crucial to the purported changes that followed. As teachers began implementing *Investigations*, they began to see a new kind of student, which in turn appeared to lead to generative changes in knowledge, beliefs and practices. As the research team embarks on a larger scale-study that will include three additional elementary schools and last at least three years, there will opportunities to test these claims and to document the kinds of changes that teachers reported.

Any model of professional development that is premised on taking into account the concerns and work conditions of teaching as well as strong subject-matter practices will necessitate a rethinking of the relationship between research on learning and teaching. There have been calls for the integration of research on learning and teaching for some time now (e.g., Romberg & Carpenter, 1986). This integration may lead to the development of models of learning through teaching, necessitating new research methods and new ideas about the origin of knowledge claims that support sound practice (Fenstermacher, 1994). To achieve systemic change, the ultimate goal is to extend models of learning to "conceive of schooling as a form of social organization that learns from and by itself" (Sabelli & Dede, 1998, p. 8).

References

Ambrose, R., Baek, J-M. & Carpenter, T. P. (in press). Children's invention of multiplication and division algorithms. To appear in xx.

Ball, D. (1990). The mathematical understandings that preservice teacher bring to teacher education. *Elementary School Journal, 90,* 449-466.

Ball, D. (1996). Teacher learning and the mathematics reforms: What we think we know and what we need to learn. *Phi Delta Kappan, xx,* 500-508.

Ball, D. & Cohen, D. (1996). xx. *Educational Researcher,* xx.

Callahan, P. & Confrey, J. (in progress). Generative domain knowledge as a framework for preservice teachers' knowledge of functions and modeling. Paper to be presented at the annual meeting of the American Educational Research Association in Montreal, Canada.

Carpenter, T. P., Fennema, E., Peterson, P., Chiang, C. & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal, 26*(4), 499 - 531.

Carpenter, T. P., Franke, M. L., Jacobs, V., Fennema, E. & Empson, S. (1998). A longitudinal study of invention and understanding in children's use of multidigit addition and subtraction procedures. *Journal for Research in Mathematics Education, 29*(1), 3-20.

Castro-Filho, J. & Confrey, J. (1999). Bringing discussions about classroom practice, curriculum, and content knowledge to teachers' daily lives. Paper to be presented at the annual meeting of the American Educational Research Association in Montreal, Canada.

Confrey, J. (in progress). Generative domain knowledge: An epistemological taxonomy for rethinking content knowledge in mathematics. Paper to be presented at the annual meeting of the American Educational Research Association in Montreal, Canada.

Duckworth, E. (1996). *"The having of wonderful ideas" and other essays on teaching and learning (second edition).* New York: Teachers College Press.

Elmore, R. (1996). Getting to scale with good educational practice. *Harvard Educational Review, 66*(1), 1-26.

Fennema, E., Carpenter, T. P., Franke, M., Levi, L., Jacobs, V. & Empson, S. B. (1996). Mathematics instruction and teachers' beliefs: A longitudinal study of using children's thinking. *Journal for Research in Mathematics Education, 27*(4), 403-434.

Fennema, E., Franke, M., Carpenter, T. P. & Carey, D. (1993). Using children's knowledge in instruction. *American Educational Research Journal, 30,* 555-583.

Fenstermacher, G. (1994). The knower and the known: The nature of knowledge in research on teaching. *Review of Research in Education, 20,* 3-56.

Floden, R. (1997). Reforms that call for teaching more than you understand. In N. Burbules & D. Hansen (Eds.), *Teaching and its predicaments* (pp. 11-28). Boulder, CO: Westview Press.

Franke, M. L., Carpenter, T. P., Levi, L. & Fennema, E. (1998). Teachers as learners: Developing understanding through children's thinking. Paper presented at the annual meeting of the American Educational Research Association in San Diego, April 1998.

Franke, M. L., Kazemi, E., Shih, J. & Biagetti, S. (1998). Cases as contexts for teacher learning. *Proceedings of the twentieth annual meeting of North American chapter of the International Group for the Psychology of Mathematics Education,* vol. 2 (pp. 675-680). Raleigh, NC: North Carolina State University.

Fullen, M. with S. Stiegelbauer (1991). *The new meaning of educational change.* New York: Teachers College Press.

Fuson, K., Wearne, D., Hiebert, J., Human, P., Murray, H., Olivier, A., Carpenter, T. P. & Fennema, E. (1997). Children's conceptual structures for multidigit numbers and methods of multidigit addition and subtraction. *Journal for Research in Mathematics Education, 28,* 130-162.

Goldsmith, L. & Schifter, D. (1997). Understanding teachers in transition: Characteristics of a model for developing teachers. In E. Fennema & B. S. Nelson (Eds.), *Mathematics teachers in transition* (pp. 19-54). Mahwah, NJ: Lawrence Erlbaum.

Hersh, R. (1997). *What is mathematics really?* New York: Oxford University Press.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Human, P., Murray, H., Olivier, A. & Wearne, D. (1996). Problem solving as a basis for reform in curriculum and instruction: The case of mathematics. *Educational Researcher, 25*(4), 12-21.

Kamii, C. (1994). *Young children continue to reinvent arithmetic: Third grade.* New York: Teachers College Press.

Kamii, C. and Dominick, A. (1998). To teach or not to teach algorithms. *Journal of Mathematical Behavior, 16* (1), 51-62.

Kennedy, M. (1990). A survey of recent literature on teachers' subject matter knowledge. (Report No. IP 90-3). East Lansing, MI: National Center for Research on Teacher Learning.

Knapp, N. & Peterson, P. (1995). Teachers' interpretations of 'cgi' after four years: Meanings and practices. *Journal for Research in Mathematics Education, 26,* 40-65.

Lave, J. (1996). Teaching, as learning, in practice. *Mind, Culture, and Activity: An International Journal, 3*(3), 149-164.

Lave, J. & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation.* New York: Cambridge.

Leinhardt, G., Putnam, R., Stein, M. K. & Baxter, J. (1991). Where subject-matter knowledge matters. In J. Brophy (Ed.), *Advances in research on teaching: Teachers' knowledge of subject matter as it relates to their teaching practice.* Greenwich, CT: JAI.

Little, J. W. (1993). Teachers' professional development in a climate of reform. *educational Evaluation and Policy Analysis, 15*(2), 129-151.

Lieberman, A. (1996). Practices that support teacher development. In M. McLaughlin & I. Oberman (Eds.), *Teacher learning: New policies, new practices* (pp. 185-201). New York: Teachers College Press.

Ma, L. (in press). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States.* Mahwah, NJ: Lawrence Erlbaum.

McLaughlin, M. W. (1993). What matters most in teachers' workplace context? In J. W. Little & M. W. McLaughlin (Eds.), *Teachers' work: Individuals, colleagues, and contexts* (pp. 79-103). New York: Teachers College Press.

McLaughlin, M. W. & Oberman, I. (1996). Introduction. In M. McLaughlin & I. Oberman (Eds.), *Teacher learning: New policies, new practices* (pp. ix-xi). New York: Teachers College Press.

Porter, A. (1989). A curriculum out of balance: The case of elementary school mathematics. *Educational Researcher, 6*(xx), 1-40.

Post, T., Harel, G., Behr, M. & Lesh, R. (1991). Intermediate teachers' knowledge of rational number concepts. In E. Fennema, T. Carpenter & S. Lamon (Eds.). *Integrating research on teaching and learning mathematics.* Albany: SUNY.

Richardson, V. (1990). Significant and worthwhile change in teaching practice. *Educational Researcher, 19*(7), 10-18.

Romberg, T. & Carpenter, T. P. (1986). Research on teaching and learning mathematics: Two disciplines of scientific inquiry. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (pp. 850 - 873). New York: Macmillan.

Sabelli, N. & Dede, C. (1998). Integrating educational research and practice: Reconceptualizing the goals and process of research to improve educational practice. (final draft).

Sarason, S. (1990). *The predictable failure of educational reform : can we change course before it's too late?* San Francisco: Jossey-Bass.

Sherin, M. (1998). Developing teachers' ability to identify student conceptions during instruction. *Proceedings of the twentieth annual meeting of North American chapter of the International Group for the Psychology of Mathematics Education,* vol. 2 (pp. 761-767). Raleigh, NC: North Carolina State University.

Simon, M. (1997). developing new models of mathematics teaching: An imperative for research on mathematics teacher development. In E. Fennema & B. S. Nelson (Eds.), *Mathematics teachers in transition* (pp. 55-86). Mahwah, NJ: Lawrence Erlbaum.

Smith, J. P. (1996). Efficacy and teaching by telling: A challenge for reform. *Journal for Research in Mathematics Education, 27,* 387-402.

Stroup, W. (in progress). xx. Paper to be presented at the annual meeting of the American Educational Research Association in Montreal, Canada.

TERC, (1995-1998). *Investigations in number, data, and space (grades K-5).* Menlo Park, CA: Dale Seymour.

Thurston, W. P. (1998). On proof and progress in mathematics. In T. Tymoczko (Ed.). *New directions in the philosophy of mathematics: An anthology (revised and expanded edition)* (pp. 337-355). Princeton, NJ: Princeton University Press.

U.S. Department of Education (1996). *Pursuing excellence: A study of U.S. eighth-grade mathematics and science teaching, learning, curriculum, and achievement in international context. Initial findings from the third international mathematics and science study (NCES 97-198).* Washington, DC: U.S. Government Printing Office.

$$
\begin{aligned}
10 \times 20 &= 200 \\
8 \times 20 &= 160 \\
10 \times 5 &= 50 \\
8 \times 5 &= 40 \\
\hline
&\ 450
\end{aligned}
$$

Figure 1. Ms. Rojas's strategy for multiplying 18 x 25.

18

12 300,
11 275
10x 250

20 500
19 25
475
18 450

Figure 2. Ms. Huston's strategies for multiplying 18 x 25 a) using a tower to build on 25 10s; and b) using clusters to compensate.

1

$$18 \times 25 = x$$

$$\begin{array}{rl} & 20 \times 25 = 500 \\ - & 2 \times 25 = 50 \\ \hline & 18 \times 25 = 450 \end{array}$$

Figure 3. A compensating strategy for multiplying 18 x 25.

$$\begin{array}{r} 144 \\ \div 2 \\ \hline 72 \\ \div 2 \\ \hline 36 \\ \div 2 \\ \hline 18 \\ \div 2 \\ \hline 9 \end{array}$$

So, $144 \div 8 = 9$

Figure 4. Fictional student's strategy for dividing 144 by 8.

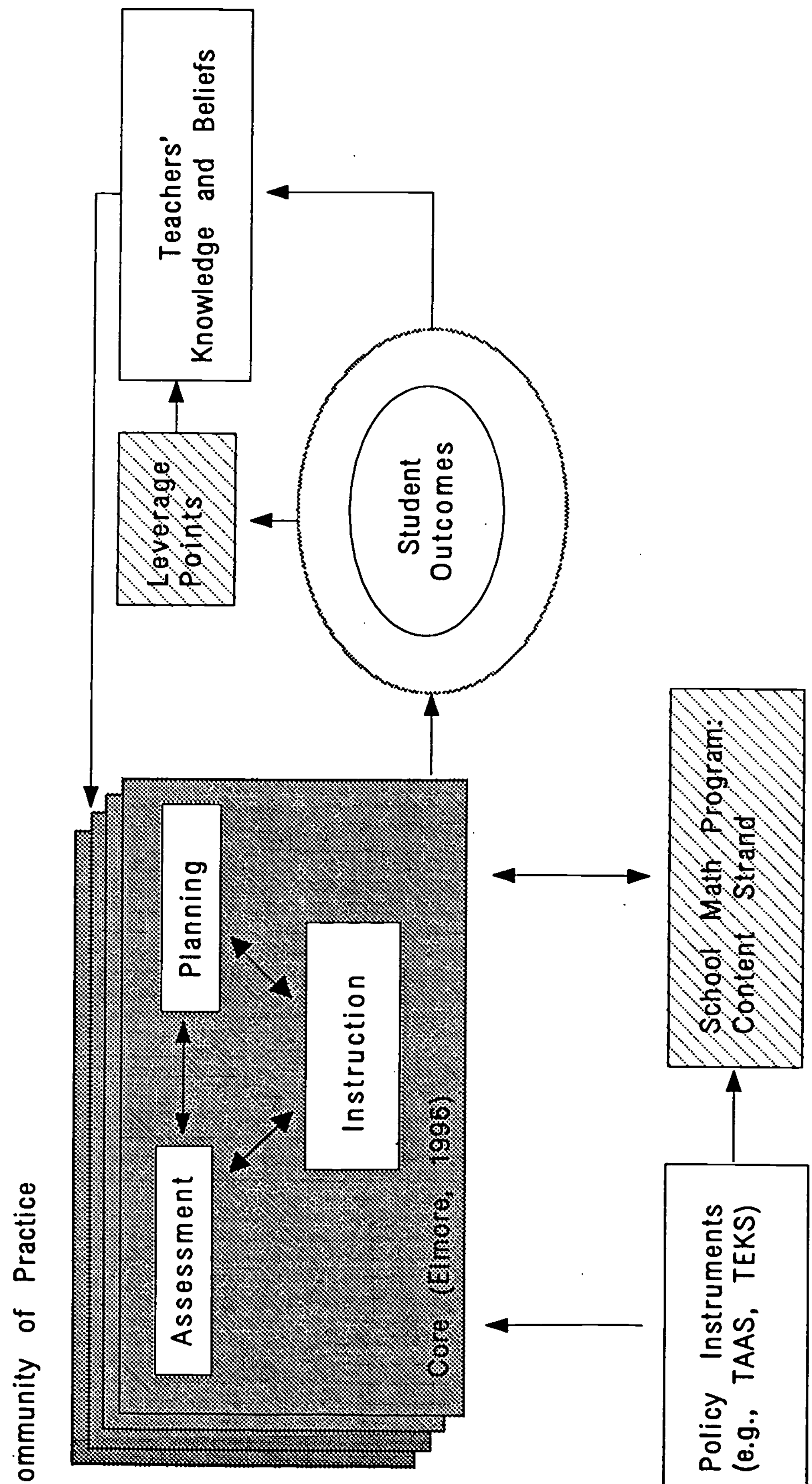

Figure 5. Proposed model of professional development

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